Problem Set 6

Problem 1

Suppose $f(x)$ and $g(x)$ are integrable on $[a, b]$ and that $g(x)$ is bounded below by $c > 0$ on $[a, b]$. It suffices to show that $\frac{1}{g(x)}$ is integrable on $[a, b]$ because then $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ is integrable. For this, first note that $\frac{1}{g(x)}$ is defined and bounded because $g(x) > c > 0$. Now, suppose $\epsilon > 0$. Because $g(x)$ is integrable, there exists $\delta > 0$ such that for all partitions $\mathcal{P}$ with $|\mathcal{P}| < \delta$,

$$|U_g(\mathcal{P}) - L_g(\mathcal{P})| < c^2\epsilon.$$

So, for all such $\mathcal{P}$,

$$\begin{aligned}
\left|U_{1/g}(\mathcal{P}) - L_{1/g}(\mathcal{P})\right| &= \left|\sum_{i=1}^{n} \left(\sup_{[\Delta x_i]} \frac{1}{g(x)} - \inf_{[\Delta x_i]} \frac{1}{g(x)}\right) \Delta x_i\right| \\
&= \left|\sum_{i=1}^{n} \left(\frac{1}{\inf_{[\Delta x_i]} g(x)} - \frac{1}{\sup_{[\Delta x_i]} g(x)}\right) \Delta x_i\right| \\
&= \left|\sum_{i=1}^{n} \left(\frac{\sup_{[\Delta x_i]} g(x) - \inf_{[\Delta x_i]} g(x)}{\inf_{[\Delta x_i]} g(x) \cdot \sup_{[\Delta x_i]} g(x)}\right) \Delta x_i\right| \\
&\leq \left|\sum_{i=1}^{n} \left(\frac{\sup_{[\Delta x_i]} g(x) - \inf_{[\Delta x_i]} g(x)}{\left(\inf_{[a,b]} g(x)\right)^2}\right) \Delta x_i\right| \\
&= \frac{1}{c^2}\left|\sum_{i=1}^{n} \left(\sup_{[\Delta x_i]} g(x) - \inf_{[\Delta x_i]} g(x)\right) \Delta x_i\right| \\
&= \frac{1}{c^2}\left|U_g(\mathcal{P}) - L_g(\mathcal{P})\right| \\
&< \epsilon.
\end{aligned}$$

This proves that $\frac{1}{g(x)}$ is integrable.

Problem 2

Let $f(x) = g(x) = x$ on the interval $[1, 3]$. These are integrable and $g(x)$ is bounded below by $c = 1 > 0$. Compute

$$\frac{\int_1^3 f(x)dx}{\int_1^3 g(x)dx} = \frac{\int_1^3 xdx}{\int_1^3 xdx} = 1 \neq 2 = \int_1^3 dx = \int_1^3 \frac{f(x)}{g(x)}dx.$$

Problem 3

Suppose $f(x)$ is integrable on $[a, b]$ and $f(x) = 0$ whenever x is rational. Suppose $\mathcal{P}$ is any partition of $[a, b]$. An interval $[\Delta x_i]$ must contain some rational number, so $\inf_{[\Delta x_i]} |f(x)| = 0$. Therefore,

$$L_{|f|}(\mathcal{P}) = \sum_{i=1}^{n} \left(\inf_{[\Delta x_i]} |f(x)|\right) \Delta x_i = 0.$$

But, $|f(x)|$ is integrable, so Corollary 19.2 says

$$\int_a^b |f(x)|dx = 0.$$

Therefore,

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx = 0$$

and

$$\int_a^b f(x)dx = 0.$$

Problem 4

Suppose $f(x)$ and $g(x)$ are integrable on $[a, b]$. First,

$$\max(f(x), g(x)) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}.$$

$f(x) - g(x)$ is integrable, so $|f(x) - g(x)|$ is integrable. Hence, $\max(f(x), g(x))$ is integrable by linearity. Furthermore, $f(x) \leq \max(f(x), g(x))$ and $g(x) \leq \max(f(x), g(x))$, so

$$\int_a^b f(x)dx \leq \int_a^b \max(f(x), g(x))dx, \qquad \text{and} \qquad \int_a^b g(x)dx \leq \int_a^b \max(f(x), g(x))dx.$$

Therefore,

$$\max\left(\int_a^b f(x)dx, \int_a^b g(x)dx\right) \leq \int_a^b \max(f(x), g(x))dx.$$

Problem 5

Let n be a positive integer, and suppose

$$\int_{-1}^1 \frac{1}{t^n} dt$$

were convergent. This implies

$$\int_0^1 \frac{1}{t^n} dt$$

is convergent. On any interval $(\epsilon, 1)$, where $0 < \epsilon < 1$, $\frac{1}{t^n}$ has anti-derivative

$$F(t) = \begin{cases} \log t & \text{if } n = 1 \\ \frac{-1}{(n-1)t^{n-1}} & \text{if } n \geq 2 \end{cases},$$

so

$$\int_0^1 \frac{1}{t^n} dt = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1}{t^n} dt = \lim_{\epsilon \to 0} (F(1) - F(\epsilon)) = \infty.$$

This is a contradiction, so

$$\int_{-1}^1 \frac{1}{t^n} dt$$

cannot be convergent.

Problem 6

(a) We claim that

$$\int_0^\infty \frac{x}{1 + x^3} dx$$

converges. First,

$$\int_0^\infty \frac{x}{1 + x^3} dx = \int_0^1 \frac{x}{1 + x^3} dx + \int_1^\infty \frac{x}{1 + x^3} dx,$$

so it suffices to show convergence on $[1, \infty)$. For this, observe that

$$\lim_{x \to \infty} \frac{\frac{x}{1+x^3}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{x^3}{1 + x^3} = 1,$$

so it suffices to show that

$$\int_1^\infty \frac{1}{x^2} dx$$

converges. Thus, compute

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{R \to \infty} \left(\frac{-1}{R} + 1\right) = 1.$$

(e) Set $u = x^2$, so that $du = 2xdx$ and

$$\int_0^\infty x^k e^{-x^2} dx = \int_0^\infty \frac{1}{2} u^{\frac{k-1}{2}} e^{-u} du.$$

This equals $\frac{1}{2}\Gamma\left(\frac{k+1}{2}\right)$ if $\frac{k+1}{2} > 0$ (i.e., $k > -1$), and diverges otherwise.

(f) We claim that

$$\int_0^1 \frac{dx}{\sqrt{x - x^3}}$$

converges. Note that the integrand is unbounded at both ends, so break it up into

$$\int_0^1 \frac{dx}{\sqrt{x - x^3}} = \int_0^c \frac{dx}{\sqrt{x - x^3}} + \int_c^1 \frac{dx}{\sqrt{x - x^3}}$$

for some $0 < c < 1$. Observe that

$$\lim_{x \to 0^+} \frac{\frac{1}{\sqrt{x - x^3}}}{\frac{1}{\sqrt{x}}} = \lim_{x \to 0^+} \sqrt{\frac{1}{1 - x^2}} = 1,$$

so to show that the first integral converges, it suffices to show that

$$\int_0^c \frac{1}{\sqrt{x}} dx$$

converges. Thus, compute

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{\sqrt{x}} dx = \lim_{\epsilon \to 0^+} (2\sqrt{1} - 2\sqrt{\epsilon}) = 2.$$

Now,

$$\lim_{x \to 1^-} \frac{\frac{1}{\sqrt{x - x^3}}}{\frac{1}{\sqrt{2}\sqrt{1-x}}} = \lim_{x \to 1^-} \sqrt{\frac{2}{x + x^2}} = 1,$$

so to show that the second integral converges, it suffices to show that

$$\int_c^1 \frac{1}{\sqrt{2}\sqrt{1 - x}} dx$$

converges. For this, use a substitution $y = 1 - x$:

$$\int_c^1 \frac{1}{\sqrt{2}\sqrt{1 - x}} dx = \frac{1}{\sqrt{2}} \int_0^{1-c} \frac{1}{\sqrt{y}} dy,$$

which we already showed converges.

(h) We claim that

$$\int_1^\infty \sin\left(\frac{1}{x}\right) dx$$

diverges. Observe that

$$\lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{\epsilon \to 0} \frac{\sin \epsilon}{\epsilon} = 1,$$

so it suffices to show that

$$\int_1^\infty \frac{dx}{x}$$

diverges. For this,

$$\int_1^\infty \frac{dx}{x} = \lim_{R \to \infty} \int_1^R \frac{dx}{x} = \lim_{R \to \infty} (\log R - \log 1) = \infty.$$

Problem 7

Let f and g be continuous and positive, and assume that $\int_0^\infty f(x)dx$ converges and that $g(x)$ is bounded. Let $M > 0$ be such that $g(x) < M$ for all x. Then, $f(x)g(x) \leq Mf(x)$ and

$$\int_0^\infty Mf(x)dx = M\int_0^\infty f(x)dx$$

converges, so

$$\int_0^\infty f(x)g(x)dx$$

converges by Theorem 21.2B.

Problem 8

Let

$$f(x) = g(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ \frac{1}{x^2} & \text{if } 1 \leq x \end{cases}.$$

Then,

$$\int_0^\infty f(x)dx = \int_0^1 \frac{dx}{\sqrt{x}} + \int_0^1 \frac{dx}{x^2} = 2 + 1 = 3$$

(see Problem 6 above). However,

$$\int_0^\infty f(x)g(x)dx = \int_0^1 \frac{dx}{x} + \int_1^\infty \frac{dx}{x^4}$$

diverges because the first integral on the right hand side diverges (see Problem 5 above).

Problem 9

Let $f(x)$ be continuous on $(0,1)$, and suppose $\int_0^1 |f(x)|^p dx$ converges for some $p > 1$. By Young's inequality,

$$|f(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{1-p},$$

so $\int_0^1 |f(x)|dx$ converges by Theorem 21.2B. This implies $\int_0^1 f(x)dx$ converges by Theorem 21.4.

Problem 10

(a) Let $u = x^2$, so that $du = 2xdx$ and

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \frac{\sin u}{2\sqrt{u}}du = \int_0^1 \frac{\sin u}{2\sqrt{u}}du + \int_1^\infty \frac{\sin u}{2\sqrt{u}}du.$$

For the first term on the right hand side, notice that when $0 < u \leq 1$,

$$0 \leq \frac{\sin u}{2\sqrt{u}} < \frac{1}{2\sqrt{u}}.$$

The integral

$$\int_0^1 \frac{1}{2\sqrt{u}}du$$

converges (see Problem 6 above), so

$$\int_0^1 \frac{\sin u}{2\sqrt{u}}du$$

converges as well by Theorem 21.2B. For the second term, we use integration by parts. Set $v = \frac{1}{2\sqrt{u}}$ and $dw = \sin u du$. Then, $dv = -\frac{1}{4u^{3/2}}du$ and $w = -\cos u$, so that

$$\int_1^\infty \frac{\sin u}{2\sqrt{u}}du = -\frac{1}{2\sqrt{u}}\cos u\Big]_1^\infty - \int_1^\infty \frac{\cos u}{4u^{3/2}}du = \frac{\cos 1}{2} - \int_1^\infty \frac{\cos u}{4u^{3/2}}du.$$

The integral on the right hand side is absolutely convergent because

$$\left|\frac{\cos u}{4u^{3/2}}\right| \le \frac{1}{4u^{3/2}}$$

and

$$\int_1^\infty \frac{1}{4u^{3/2}} du$$

converges. This proves that $\int_0^\infty \sin(x^2)dx$ converges.

(b) First, note that when $n \ge 0$ is an integer and $\frac{\pi}{4} + n\pi \le u \le \frac{3\pi}{4} + n\pi$,

$$\left|\frac{\sin u}{2\sqrt{u}}\right| \ge \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{u}} \ge \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{3\pi/4 + n\pi}}.$$

Therefore, we can bound the integral below by restricting to all such intervals:

$$\begin{aligned}\int_0^{3\pi/4+N\pi} \left|\frac{\sin u}{2\sqrt{u}}\right| du &\ge \sum_{n=0}^{N} \int_{\pi/4+n\pi}^{3\pi/4+n\pi} \left|\frac{\sin u}{2\sqrt{u}}\right| du \\ &\ge \sum_{n=0}^{N} \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{3\pi/4+n\pi}} \int_{\pi/4+n\pi}^{3\pi/4+n\pi} du \\ &= \frac{\sqrt{\pi}}{4\sqrt{2}} \sum_{n=0}^{N} \frac{1}{\sqrt{3/4+n}}.\end{aligned}$$

As N goes to infinity, this sum diverges, so $\int_0^\infty \left|\frac{\sin u}{2\sqrt{u}}\right| du$ diverges as well.

Problem 11

Suppose $\lim_{x\to\infty} f(x) = 0$, $\int_a^\infty f'(x)dx$ is absolutely convergent, and $f'(x)$ is continuous for $x \ge a$. To show that

$$\int_a^\infty f(x)\sin x dx$$

converges, we use integration by parts. Let $u = f(x)$ and $dv = \sin x dx$, so that $du = f'(x)dx$ and $v = -\cos x$. Thus,

$$\int_a^R f(x)\sin x dx = -f(x)\cos x\Big]_a^R + \int_a^R f'(u)\cos u du = f(a)\cos a - f(R)\cos R + \int_a^R f'(u)\cos u du. \quad (1)$$

First,

$$0 \le \left|\lim_{R\to\infty} f(R)\cos R\right| = \lim_{R\to\infty} |f(R)\cos R| \le \lim_{R\to\infty} |f(R)| = \left|\lim_{R\to\infty} f(R)\right| = 0,$$

so $\lim_{R\to\infty} f(R)\cos R = 0$. Second,

$$\lim_{R\to\infty} \int_a^R |f'(u)\cos u| du \le \lim_{R\to\infty} \int_a^R |f'(u)| du < \infty,$$

so $\int_a^\infty f'(u)\cos u du$ is absolutely convergent (and therefore convergent). Thus, taking R to infinity in equation (1) shows $\int_a^\infty f(x)\sin x dx$ converges.